The Alpha Ceiling and Twin Prime Distribution

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17 September 2025

# Abstract

We investigate a geometric formulation of prime distribution based on prime triangles and their associated alpha angle. A proved inequality shows that a non-twin prime triangle can exceed the alpha angle of a twin only after a computable threshold, which for the minimal gap occurs beyond 2*pn​*. We call this bound the alpha ceiling. Thus, any violation requires a prime beyond the threshold. Computations up to 25×109show no such violations: the ceiling remains unbroken. We conclude with two conjectures: that twin primes set local α-ceilings, and that the gap between consecutive twin primes never exceeds the smaller twin—yielding a simple uniform bound on twin-prime gaps.

# 1. Definitions

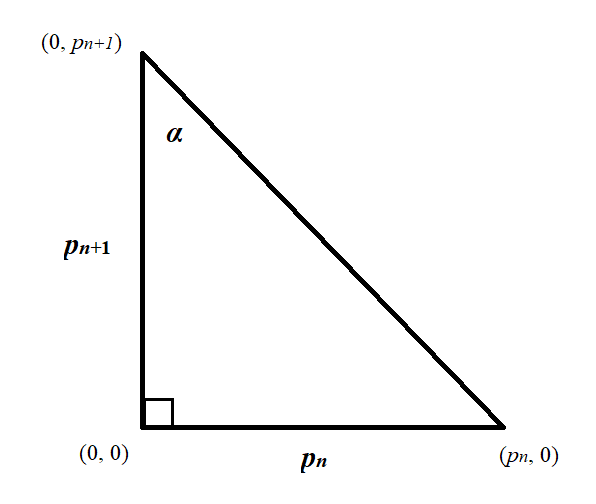
**Consecutive Primes**: Two primes (*pn​*, *pn*+1​) with no primes between them.

**Twin Primes**: Consecutive primes with gap 2, i.e. (*pn​*, *pn* ​+ 2).

**Prime Triangle**: For consecutive primes (*pn​*, *pn*+1​), define the right triangle with vertices (0, 0), (*pn*, 0), (0, *pn*+1​). As shown in Figure 1.

**Alpha Angle** (*α*): The angle at (0, *pn*+1​), opposite the leg of length *pn* of a prime triangle: *α* = arctan(*pn* / *pn​​*+1). As shown in Figure 1.

**Figure 1**: The Prime Triangle associated with consecutive primes (*pn​*, *pn*+1​).

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# 2. Lemma: The Alpha Ceiling Inequality

**Lemma**.  
Let (*pn​*, *pn​* + 2) be a twin prime pair. For any consecutive prime pair (*pk​*, *pk ​*+ *gk​*) after it, if *αk* > *αn*, then

*pk* > (*pn*)(*gk*)/2.

**Corollary (Contrapositive)***.*  
Let (*pn​*, *pn*+1​) be a twin prime pair and suppose *pn* *<* *pk <* 2*pn*. Then the consecutive-prime angle satisfies

*αk* ≤ *αn*​​.

In words: any non-twin consecutive prime pair occurring strictly between a twin *pn* and its double cannot exceed the α-ceiling set by that twin.

**Proof**.  
Since arctan is increasing, *αk* > *αn* if (*pk* / (*pk* + *gk*)) > (*pn* / (*pn*+ 2)). Cross-multiplying and simplifying yields 2*pk* > (*pn*)(*gk*), or equivalently *pk* > (*pn*)(*gk*)/2.

Figure 2 illustrates how the twin prime α sets the ceiling.

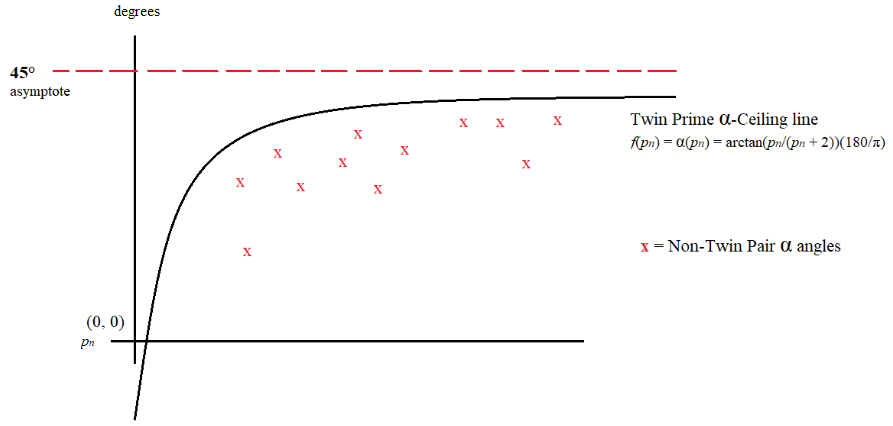
**Examples**:

• If *gk* = 4, violation requires *pk* > 2*pn*.

• If *gk* = 6, violation requires *pk* > 3*pn*.

• Larger gaps push the threshold higher.

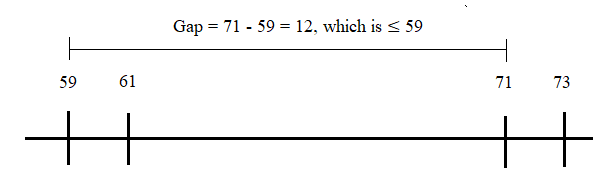
**Figure 2**: The α-ceiling: twin prime α’s set the maximum until a threshold is passed



# 3. Computational Evidence and Numeric Implications

The violation thresholds from the *α*-ceiling lemma suggest a natural bound on gaps between consecutive twin primes. Specifically, for any twin prime pair (*pn​*, *pn​* + 2), the earliest possible *α*-ceiling violation for a non-twin gap *g* > 2 occurs at *pk* > (*pn*)(*g*)/2. Applying this pattern to the minimal gap *g* = 2 implies that the next twin prime should occur **before a gap exceeding *pn*​**. Figure 3 illustrates the twin prime gap and condition.

**Figure 3**: Example of the twin-to-twin gap condition, *pm* − *pn​* ≤ *pn​*.



Computations up to 25×109 confirm the following:

* No *α*-ceiling violations were observed.
* For all consecutive twin prime pairs (*pn​*, *pn​* + 2) and (*pm*, *pm* + 2), the gap *pm* − *pn​* does not exceed *pn​*.

These results support the numeric consequence suggested by the *α*-ceiling framework.

# 4. Conjectures

**Conjecture 1 − The Twin Prime *α*-Ceiling.**  
For every twin prime pair (*pn*, *pn* + 2) and the next twin pair (*pm*, *pm* + 2), all consecutive primes (*pk*, *pk*+1) with *n < k < m* satisfy

*αk* ≤ *αn*.

Equivalently, twin primes set an *α*-ceiling that is not broken by intervening consecutive primes.

**Logical gap from Lemma to Conjecture 1.**

The lemma (and its corollary) prove only a local restriction: no consecutive prime pair with base less than 2*pn*​ can exceed *αn*. Conjecture 1 should therefore be read as a conjectural extrapolation of the lemma, supported by extensive computation but not deduced from it.

**Conjecture 2 − The Twin Prime Gap Bound.**For any twin prime pair (*pn*, *pn* + 2) with *pn* > 5, let (*pm*, *pm* + 2) be the next twin prime pair. Then

*pm* − *pn* ≤ *pn*.

Equivalently, the distance between consecutive twin primes never exceeds the smaller twin’s first prime for *pn* > 5.

*This conjecture asserts a uniform upper bound on twin-prime gaps, a sharper distributional claim than Conjecture 1 and the central novelty of this note.*

# 5. Discussion

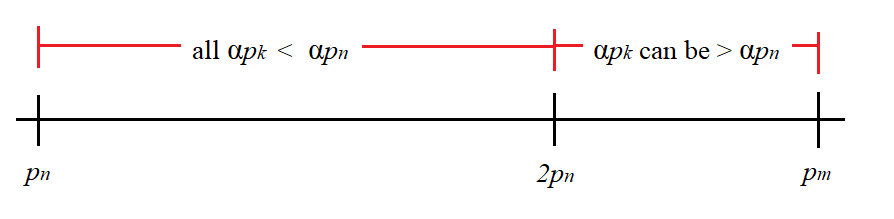
**Narrative explanation.**

The relationship between the lemma and the two conjectures can be viewed as a stepwise narrowing of possibilities.

1. The Lemma.

The α-ceiling lemma is derived in the broadest case, which allows the possibility that the next twin prime occurs after 2*pn*. Beyond this point it becomes possible, in principle, for some non-twin angle *αk* to exceed the twin’s angle *αn*.

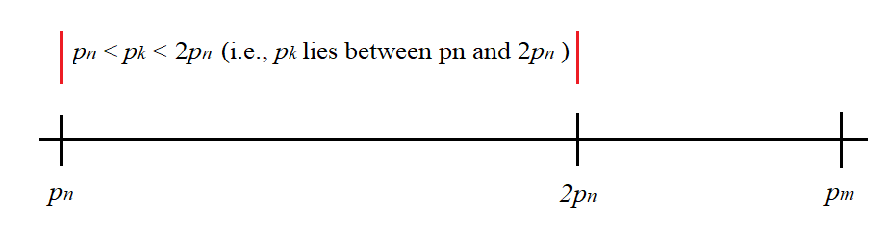
**Figure 4**: Number line with pn, 2pn, and pm , showing the zone where a violation could occur.



2. The Twin Prime *α*-Ceiling.

Conjecture 1 asserts that the ceiling is never broken: no such *αk* occurs. Equivalently, there are no consecutive primes with bases between 2*pn,* and *pm*.

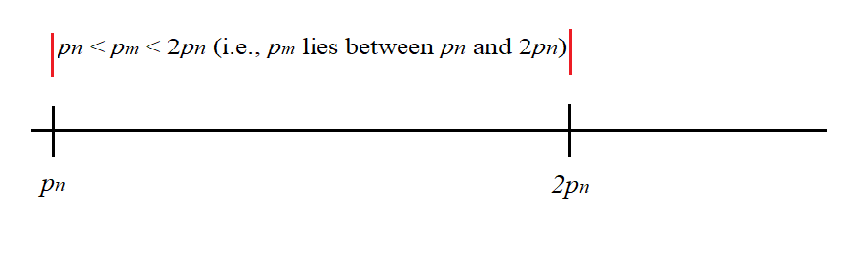
**Figure 5**: Number line showing pn, 2pn, and pm with the interval (2pn, pm) shown as empty.



3. The Twin Prime Gap Bound.

Conjecture 2 strengthens this picture by proposing that in fact *pm ≤*2*pn*. Thus, the broader case envisioned in Conjecture 1 (where the next twin lies beyond 2*pn*) does not occur.

**Figure 6**: Number line with pn and 2pn, showing pm always inside that bound.



Together, these steps clarify the relationship: the lemma allows a post-2*pn* violation, Conjecture 1 rules out such violations, and Conjecture 2 goes further by ruling out the entire scenario, keeping the next twin strictly within the 2*pn* threshold.

In summary, Conjecture 1 frames the broad ceiling principle, while Conjecture 2 sharpens it into a concrete gap bound; the former motivates the framework, but the latter is the central conjectural advance of this note.

**Summary points.**

• The *α*-ceiling lemma provides a clear inequality: violation requires *pk* > (*pn*)(*gk*)/2.

• Applying this pattern to the minimal gap *g* = 2 leads naturally to the bound expressed in Conjecture 2.

• While these conjectures do not prove the infinitude of twin primes, they offer a geometric and numeric framework connecting prime triangles, *α*-angles, and twin-prime distribution.

• Computations up to 25×109 support both conjectures: *α*-ceilings remain unbroken, and twin-prime gaps never exceed the previous twin’s first prime.

# References

Hardy, G. H., & Wright, E. M. (1979). An Introduction to the Theory of Numbers. Oxford University Press.

James Grime, *Prime Spirals*, Numberphile, [interview with Brady Haran], https://youtu.be/iFuR97YcSLM?si=UZ64UWcOWxIQ\_\_wZ